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REGULAR SEMIGROUPS WITH MULTIPLICATIVE ORTHODOX TRANSVERSALS

PEI WANG and XIANGJUN KONG

School of Mathematical Sciences
Qufu Normal University
Qufu, Shandong, 273165
P. R. China
e-mail: wangpei9778@163.com

Abstract

In this paper, the concept of multiplicative orthodox transversals is first introduced, and some properties on multiplicative orthodox transversals are investigated. Then a construction of a regular semigroup with a multiplicative orthodox transversal is established. Finally, two examples are given to show a regular non-orthodox and an orthodox semigroup with a multiplicative orthodox transversal, respectively.

1. Introduction

The concept of an inverse transversal of a regular semigroup was first introduced by Blyth and McFadden in 1982 [1]. Afterwards, this class of regular semigroups attracted several author's attention and a series of important results were obtained [1, 2, 7, 8]. If S is a regular semigroup, then an inverse transversal of S is an inverse subsemigroup S^o such that S^o meets $V(a)$, precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S : axa = a \text{ and } xax = x\}$ denotes the set of inverses

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of a. The intersection of $V(a)$ and S^o is denoted by $V_{S^o}(a)$ and the unique element of $V_{S^o}(a)$ is denoted by a^o. It is well known that, the sets $I = \{e \in S : ee^o = e\}$ and $\Lambda = \{f \in S : f^o f = f\}$ are left regular and right regular bands, respectively, and play an important role in the study of regular semi groups with inverse transversals. Orthodox transversals were introduced by Chen [3] as a generalization of inverse transversals, and an excellent structure theorem for regular semi groups with quasi-ideal orthodox transversals was also given. Afterwards, Chen and Guo [4] considered the general case of orthodox transversals and investigated some properties concerning the sets I and Λ (defined below). The second author [6] constructed regular semigroups with quasi-ideal orthodox transversals by a formal set. Analogous to the inverse transversal, the concept of a multiplicative orthodox transversal can be introduced. The main purpose of this paper is to introduce and study the class of regular semigroups with multiplicative orthodox transversals.

Let S be a semigroup and S^o be a subsemigroup of S. As usual, we shall write the set $S^o \cap V(x)$ as $V_{S^o}(x)$ for every $x \in S$, x^o as an element of $V_{S^o}(x)$, and

$$I = \{aa^o : a \in S,\ a^o \in V_{S^o}(a)\},\ \Lambda = \{a^o a : a \in S,\ a^o \in V_{S^o}(a)\}.$$

Then S^o is said to be an *orthodox transversal* of S, if the following conditions are satisfied.

(1.1) For all $a \in S$, $V_{S^o}(a) \neq \emptyset$;

(1.2) If $\forall a, b \in S$, and $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Note that, if S^o is an orthodox transversal of S, then S is a regular semigroup by (1.1) and S^o is an orthodox subsemigroup of S by (1.2).

A subsemigroup S^o of S is said to be a *quasi-ideal* of S, if $S^o S S^o \subseteq S^o$.

Finally, the following theorem will be frequently used without further mention.

(1.3) Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;

(1.4) Let a, b be elements of a semigroup S. Then $ab \in R_a \cap L_b$, if and only if $L_a \cap R_b$ contains an idempotent.

We adopt the terminology and notation of [3, 6, 7].

2. Multiplicative Orthodox Transversals

Definition 2.1. Let S be a regular semigroup with an orthodox transversal S^o. Then we shall say S^o is multiplicative, if

$$(\forall x, y \in S)(\forall x^o \in V_{S^o}(x), y^o \in V_{S^o}(y))\ x^o xyy^o \in E(S^o),$$

where $E(S^o)$ is the band of idempotents of S^o.

It is well-known that, the sets I and Λ play an important role in both the inverse transversal and the orthodox transversal. By I and Λ, an orthodox transversal S^o is said to be *multiplicative* and a *quasi-ideal* can be equivalently defined. An orthodox transversal S^o is

(1) multiplicative, if $\Lambda I \subseteq E(S^o)$.

(2) a quasi-ideal, if $\Lambda I \subseteq S^o$.

By Lemma 2.1 in [7], we have

Lemma 2.2. *Let S be a regular semigroup with a multiplicative orthodox transversal S^o. Then S^o is a quasi-ideal of S.*

Theorem 2.3. *Let S be a regular semigroup with an orthodox transversal S^o. If S^o is a band, then S^o is multiplicative, if and only if S^o is a quasi-ideal.*

Proof. By Lemma 2.2, the condition is clearly necessary. Conversely, if S^o is a quasi-ideal, then $\Lambda I \subseteq S^o$. So, $\Lambda I \subseteq E(S^o)$ since we also have $E(S^o) = S^o$ by S^o is a band. That is, S^o is multiplicative.

If S is a regular semigroup with an inverse transversal S^o, Green's relations $\mathcal{L}$ and $\mathcal{R}$ are given by

$$(x, y) \in \mathcal{L} \Leftrightarrow x^o x = y^o y, \quad (x, y) \in \mathcal{R} \Leftrightarrow xx^o = yy^o.$$

As for an orthodox transversal S^o, we have to get a weaker result.

Theorem 2.4. *Green's relations $\mathcal{L}$ and $\mathcal{R}$ are given by*

$$(x, y) \in \mathcal{L} \Leftrightarrow V_{S^o}(x)x = V_{S^o}(y)y, \quad (x, y) \in \mathcal{R} \Leftrightarrow xV_{S^o}(x) = yV_{S^o}(y).$$

Proof. If $V_{S^o}(x)x = V_{S^o}(y)y$, then for any $x^o \in V_{S^o}(x)$, there exists $y^o \in V_{S^o}(y)$ such that $x^o x = y^o y$. So $x\mathcal{L}x^o x = y^o y\mathcal{L}y$. If $(x, y) \in \mathcal{L}$, then for any $x^o \in V_{S^o}(x)$ and $y^o \in V_{S^o}(y)$, we have $y\mathcal{R}yy^o$ and $y\mathcal{L}x\mathcal{L}x^o x$. Since yy^o and $x^o x$ are idempotents, it follows from (1.3) that there exists an inverse y' in $L_{yy^o} \cap R_{x^o x}$ of y such that $x^o x = y'y$. Since $x^o\mathcal{R}x^o x\mathcal{R}y'\mathcal{L}yy^o\mathcal{L}y^o$, it following from Theorem 2.4 in [4] that $y' \in S^o$. That is, to say for any $x^o \in V_{S^o}(x)$, there exists $y' \in V_{S^o}(y)$ such that $x^o x = y'y$, and thus, $V_{S^o}(x)x \subseteq V_{S^o}(y)y$. Similarly, we have $V_{S^o}(y)y \subseteq V_{S^o}(x)x$. Therefore, $V_{S^o}(x)x = V_{S^o}(y)y$. The dual result can be proved similarly.

Theorem 2.5. *Let S be a regular semigroup with an orthodox transversal S^o. Suppose that $V_{S^o}(e) \subseteq E(S^o)$ for every $e \in E(S)$. If $x' \in V(x)$ for any $x \in S$, then for any $x^o \in V_{S^o}(x)$, there exists $x^{oo} \in V_{S^o}(x^o)$, such that*

$$x^{oo}x^o \in V_{S^o}(xx') = V_{S^o}(xx^o),\ x^o x^{oo} \in V_{S^o}(x'x) = V_{S^o}(x^o x),$$

and

$$x^{oo} \in V_{S^o}(x') = V_{S^o}(x'xx^o) = V_{S^o}(x^o xx') = V_{S^o}(x^o).$$

Proof. Let $x \in S$ and $x' \in V(x)$. Since $xx'\mathcal{R}x$, it follows from $xx' \cdot V_{S^o}(xx') = x \cdot V_{S^o}(x)$ that there exists $(xx')^o \in V_{S^o}(xx')$ such that $xx' \cdot (xx')^o = xx^o$. By the assumption, $(xx')^o \in E(S^o)$. So that

$$xx^o(xx')^o = xx'(xx')^o(xx')^o = xx'(xx')^o = xx^o,$$

and

$$(xx')^o xx^o = (xx')^o xx'(xx')^o = (xx')^o,$$

which shows that $xx^o\mathcal{L}(xx')^o$. So $x^o\mathcal{L}(xx')^o$ since $x^o\mathcal{L}xx^o$. Since $(xx')^o \in E(S^o)$, so for $(xx')^o \in V_{S^o}((xx')^o)$, there exists $x^{oo} \in V_{S^o}(x^o)$ such that $x^{oo}x^o = (xx')^o(xx')^o = (xx')^o$, that is, $x^{oo}x^o \in V_{S^o}(xx')$. And since, we also have $x^{oo}x^o \in V_{S^o}(xx^o)$, so $x^{oo}x^o \in V_{S^o}(xx') = V_{S^o}(xx^o)$. Similarly, there exists $x^{o'} \in V_{S^o}(x^o)$ such that $x^o x^{o'} \in V_{S^o}(x'x) = V_{S^o}(x^o x)$. Since, we also have $x^o x^{oo} \in V_{S^o}(x^o x)$, so $x^o x^{oo} \in V_{S^o}(x'x) = V_{S^o}(x^o x)$.

By using the above facts, we have

$$\begin{aligned} x^{oo}x'x^{oo} &= x^{oo}x'xx'x^{oo}x^o x^{oo} = x^{oo}x'xx' \cdot (xx')^o xx^o \\ &= x^{oo}x'xx^o x^{oo} = x^{oo} \cdot x^o x^{oo}x'x \cdot x^o x^{oo} \\ &= x^{oo}x^o x^{oo} = x^{oo}, \end{aligned}$$

and

$$\begin{aligned} x'x^{oo}x' &= x'xx'x^{oo}x^o x^{oo}x' = x'xx'(xx')^o x^{oo}x' \\ &= x'xx^o x^{oo}x' = x'x(x'x)^o x' = x'. \end{aligned}$$

Consequently, $x^{oo} \in V_{S^o}(x')$. Also, we have

$$x'xx^o \cdot x^{oo} \cdot x'xx^o = x'x(x'x)^o x'x \cdot x^o = x'xx^o,$$

and

$$x^{oo} \cdot x'xx^o \cdot x^{oo} = x^{oo} \cdot xx^{oo} \cdot x'x \cdot x^o x^{oo} = x^{oo} \cdot x^o x^{oo} = x^{oo}.$$

That is to say $x^{oo} \in V_{S^o}(x'xx^o)$. Similarly, $x^{oo} \in V_{S^o}(x^o xx')$. Therefore,

$$x^{oo} \in V_{S^o}(x') = V_{S^o}(x'xx^o) = V_{S^o}(x^o xx') = V_{S^o}(x^o).$$

Theorem 2.6. *Let S be a regular semigroup with an orthodox transversal S^o. Suppose that $V_{S^o}(e) \subseteq E(S^o)$ for every $e \in E(S)$. Then for any $x, y \in S$, $V(x) \cap V(y) \neq \emptyset$, if and only if $V_{S^o}(x) = V_{S^o}(y)$.*

Proof. Take $a' \in V(x) \cap V(y)$. Then by Theorem 2.5, for any $x^o \in V_{S^o}(x)$, $y^o \in V_{S^o}(y)$, there exist $x^{oo} \in V_{S^o}(x^o)$, $y^{oo} \in V_{S^o}(y^o)$, such that $x^{oo} \in V_{S^o}(a')$, $y^{oo} \in V_{S^o}(a')$. Thus, $V_{S^o}(x^o) = V_{S^o}(a') = V_{S^o}(y^o)$. Take $x^{oo} \in V_{S^o}(x^o) = V_{S^o}(y^o)$. Then

$$x^o \in V_{S^o}(x) \cap V_{S^o}(x^{oo}) \text{ implies } V_{S^o}(x) = V_{S^o}(x^{oo}),$$

$$y^o \in V_{S^o}(y) \cap V_{S^o}(x^{oo}) \text{ implies } V_{S^o}(y) = V_{S^o}(x^{oo}),$$

and therefore, $V_{S^o}(x) = V_{S^o}(y)$.

The converse is obvious.

Theorem 2.7. *Let S be a regular semigroup with an orthodox transversal S^o. Then S^o is multiplicative, if and only if S^o is a quasi-ideal of S and $V_{S^o}(e) \subseteq E(S^o)$ for every $e \in E(S)$.*

Proof. If S^o is multiplicative, then by Lemma 2.2, S^o is a quasi-ideal of S and for any $e \in E(S)$, $e^o \in V_{S^o}(e)$, we have $e^o = e^o e e^o = e^o e e e^o \in E(S^o)$. Conversely, let $e \in \Lambda$, $f \in I$, and let $x \in V_{S^o}(ef)$. Then $(fxe)^2 = fxe$ and $ef \in V(fxe) \cap S^o$, so by assumption, we have that $ef \in E(S^o)$. Thus $\Lambda I \subseteq E(S^o)$, and consequently, S^o is multiplicative.

Let S be a regular semigroup with a multiplicative inverse transversal S^o, and let $\langle E(S) \rangle$ be the idempotent-generated regular subsemigroup of S. In [1], it has been shown that $\langle E(S) \rangle = \{x \in S : x^o \in E(S^o)\}$. As for the orthodox transversal, we have a similar result.

Theorem 2.8. *Let S be a regular semigroup with a multiplicative orthodox transversal S^o. Then $\langle E(S) \rangle = \{x \in S : V_{S^o}(x) \subseteq E(S^o)\}$.*

Proof. For any $x \in S$, if $V_{S^o}(x) \subseteq E(S^o)$, take $x^o \in V_{S^o}(x)$, then we have

$$x = xx^o x = xx^o \cdot x^o x \in \langle E(S) \rangle.$$

For the converse inclusion, it suffices to prove that $V_{S^o}(xy) \subseteq E(S^o)$ for all $x, y \in E(S)$. Let $x^o \in V_{S^o}(x)$ and $y^o \in V_{S^o}(y)$, then by Theorem 2.7, $x^o, y^o \in E(S^o)$. Since S^o is a multiplicative orthodox transversal of S, we have $x^o xyy^o \in E(S^o)$, and so $y^o x^o xyy^o x^o \in E(S^o)$. But $y^o x^o xyy^o x^o \in V(xy)$, so we have that $V_{S^o}(xy)$ contains an idempotent, so by Corollary 2.1 of [4], it consists of idempotents only. Now, the proof of this theorem is completed.

3. The Structure Theorem

In 1999, Chen [3] gave a structure theorem for regular semigroups with quasi-ideal orthodox transversals. There a regular mapping $*$ means

a mapping $\Lambda \times I \to S^o$, denoted by $(\lambda, i) \to \lambda * i$, which satisfies three conditions:

(a) $(\forall e, f \in E^o)\ e(\lambda * i)f = (e\lambda) * (if)$.

(b) If $\lambda \in E^o$ or $i \in E^o$, then $\lambda * i = \lambda i$.

(c) If $\lambda^+, i^* \in E^o$ and $\lambda^+ \mathcal{R} \lambda \mathcal{L} \lambda'$ and $i^* \mathcal{L} i \mathcal{R} i'$, then $(i^* i') \cdot V_{S^o}(\lambda' * i') \cdot (\lambda' \lambda^+) \subseteq V_{S^o}(\lambda * i)$.

If the mapping $*$ is $\Lambda \times I \to E^o$ instead of $\Lambda \times I \to S^o$, then we can construct a regular semigroup with a multiplicative orthodox transversal, and the condition (c) can be removed. Since, this construction is a modification of that Chen gave in [3], the proofs have a good deal in common. So, we only give the outline of the proof, for more details, see [3].

In what following, S^o denotes an orthodox semigroup, E^o denotes the set of the set of idempotents of S^o, and σ denotes the minimum inverse semigroup congruence on S^o. For $a \in S^o$, the idempotents in $L_a(R_a)$ are denoted by $a_*, a_{**}, \cdots (a_+, a_{++}, \cdots)$, the σ-class containing a will be denoted by $T(a)$. Let I and Λ are bands such that $I \cap \Lambda = E^o$. The triple (I, S^o, Λ) is said to be *permissible*, if E^o is a quasi-ideal orthodox transversal of both I and Λ, and $L(I) = I$, $R(\Lambda) = \Lambda$. Denote the elements of I by $i, i_1, i', \cdots$, and the $\mathcal{R}$-class of I containing i by R_i. Denote the elements of Λ by $\lambda, \lambda_1, \lambda' \cdots$, and the $\mathcal{L}$-class of Λ containing λ by L_λ.

Theorem 3.1. *Suppose that* (I, S^o, Λ) *is a permissible triple with a mapping* $* : \Lambda * I \to E^o$, *denoted by* $(\lambda, i) \to \lambda * i$, *which satisfies the following*:

(3.1) $(\forall e, f \in E^o)\ e(\lambda * i)f = (e\lambda) * (if)$,

(3.2) *If* $\lambda \in E^o$ *or* $i \in E^o$, *then* $\lambda * i = \lambda i$.

We define a subset of $I / \mathcal{R} \times S^o / \sigma \times \Lambda / \mathcal{L}$ as follows:

$$\Gamma = \{(R_i, T(x), L_\lambda) : (\exists i^*, \lambda^+ \in E^o)\ i\mathcal{L}i^*\mathcal{R}x,\ \lambda\mathcal{R}\lambda^+\mathcal{L}x\},$$

where $x \in S^o$. Define multiplication in Γ by

$$(R_i, T(x), L_\lambda)(R_{i_1}, T(x_1), L_{\lambda_1}) = (R_{ia_+}, T(a), L_{a_*\lambda_1}),$$

where $a = x(\lambda * i_1)x_1$. Then Γ is a regular semigroup containing a multiplicative orthodox transversal isomorphic to S^o.

Conversely, every regular semigroup with a multiplicative orthodox transversal can be constructed in this way.

Proof. Since condition (c) is crucial to the proof of that the multiplication in Γ is well-defined [3], we first proved that (c) is valid under our assumption.

If $\lambda^+, \lambda^* \in E^o$ and $\lambda^+\mathcal{R}\lambda\mathcal{L}\lambda'$ and $i^*\mathcal{L}i\mathcal{R}i'$, take $s \in V_{S^o}(\lambda' * i')$. Then, we have

$$\begin{aligned}(\lambda * i)(i^*i's\lambda'\lambda^+)(\lambda * i) &= (\lambda * ii^*i')s(\lambda'\lambda^+\lambda * i)\\ &= (\lambda * i')s(\lambda' * i)\\ &= (\lambda\lambda' * i')s(\lambda' * i'i)\\ &= \lambda(\lambda' * i')s(\lambda' * i')i\\ &= \lambda(\lambda' * i')i\\ &= \lambda * i,\end{aligned}$$

and

$$\begin{aligned}(i^*i's\lambda'\lambda^+)(\lambda * i)(i^*i's\lambda'\lambda^+) &= i^*i's(\lambda'\lambda^+\lambda * ii^*i')s\lambda'\lambda^+\\ &= i^*i's(\lambda' * i')s\lambda'\lambda^+\\ &= i^*i's\lambda'\lambda^+.\end{aligned}$$

Thus

$$(i^* i') \cdot V_{S^o}(\lambda' * i') \cdot (\lambda' \lambda^+) \subseteq V_{S^o}(\lambda * i).$$

Up to now, all the conditions of Theorem 3.4 in [3] are satisfied. Therefore, as the proofs in [3], Γ is a regular semigroup containing a quasi-ideal orthodox transversal W, where

$$W = \{(R_i, T(x), L_\lambda) : (\exists \overline{x} \in T(x), i' \in R_i \cap E^o, \lambda' \in L_\lambda \cap E^o) \lambda' \mathcal{L} \overline{x} \mathcal{R} i'\}.$$

In the following, we only need to prove that W is a multiplicative orthodox transversal of Γ.

Let $(R_i, T(x), L_\lambda) \in W$, by Lemma 3.7 in [3] , we can assume that $i, \lambda \in E^o$. Then $(R_i, T(x), L_\lambda) \in E(W)$, if and only if $x = x\lambda ix$. In fact,

$$(R_i, T(x), L_\lambda)^2 = (R_{i(x\lambda ix)_+}, T(x\lambda ix), L_{(x\lambda ix)_* \lambda}).$$

If $x = x\lambda ix$, then $T(x) = T(x\lambda ix)$, and $R_{i(x\lambda ix)_+} = R_{ix_+} = R_{x_+} = R_o$, since $i\mathcal{R}x\mathcal{R}_{x_+}$. Similarly, $L_{(x\lambda ix)_* \lambda} = L_\lambda$. That is, $(R_i, T(x), L_\lambda) \in E(W)$. Conversely, if $(R_i, T(x), L_\lambda) \in E(W)$, then $T(x) = T(x\lambda ix)$. Let x' be the common inverse of x and $x\lambda ix$. Then $x'x\lambda ixx' = x'$, hence $xx'x\lambda ixx'x = xx'x$, and consequently, $x\lambda ix = x$.

Take

$$k = (R_i, T(x), L_\lambda), l = (R_j, T(y), L_\mu) \in \Gamma,$$

and

$$(R_{i'}, T(x'), L_{\lambda'}) \in V_W(k), (R_{j'}, T(y'), L_{\mu'}) \in V_W(l),$$

where $T(x'), T(y')$ are the inverses of $T(x), T(y)$ in S^o / σ, respectively, and $V_W(a)$ as described in Lemma 3.8 in [3]. We can assume that $i', \lambda', j', \mu' \in E^o$. Then

$$(R_{i'}, T(x'), L_{\lambda'})(R_i, T(x), L_\lambda)(R_j, T(y), L_\mu)(R_{j'}, T(y'), L_{\mu'})$$

$$= (R_{i'a_+}, T(a), L_{a*\lambda})(R_{jb_+}, T(b), L_{b*\mu'})$$

$$= (R_{i'a_+c_+}, T(c), L_{c_*b_*\mu'}) \in W,$$

where $a = x'(\lambda' * i)x = x'\lambda' ix$, $b = y(\mu * j')y' = y\mu j'y'$, $c = a(a * \lambda * jb_+)$ $b = a(\lambda * j)b$.

Also, we have

$$c \cdot c_*b_*\mu' \cdot i'a_+c_+ \cdot c$$

$$= cb_*\mu' i'a_+c$$

$$= a(\lambda * j)bb_*mu'i'a_+ \cdot a(\lambda * j)b$$

$$= x'\lambda' ix(\lambda * j)y\mu j'y'\mu' i'x'\lambda' ix(\lambda * j)y\mu j'y'$$

$$= x'\lambda' ix(\lambda * j)y\mu j'y' \cdot x'\lambda' ix(\lambda * j)y\mu j'y'$$

$$= c \cdot c. \qquad (\text{ since } u', i' \in E^o \text{ and } i'\mathcal{R}x', \mu'\mathcal{L}y')$$

Thus $x'\lambda' ix \subseteq x'E^o x \subseteq E^o$, since $x, x' \in S^o$ and S^o is orthodox. Similarly, $y\mu j'y' \in E^o$. Consequently, $c = x'\lambda' ix \cdot (\lambda * j) \cdot y\mu j'y' \in E^o \cdot E^o \cdot E^o \subseteq E^o$ since S^o is orthodox. Therefore, $V_W(k)klV_W(l) \subseteq E(W)$, and so W is a multiplicative orthodox transversal of Γ.

Conversely, let S be a regular semigroup and S^o be a multiplicative orthodox transversal of S. For every $(\lambda, i) \in \Lambda \times I$, put $\lambda * i = \lambda i$. Then, $\lambda * i \in E^o$ since S^o is a multiplicative orthodox transversal of S. All the other conditions in Theorem 3.1 are satisfied as in [3].

In the following, we will give two examples, both of which are based on 2×2 matrices. In Example 2, we take use of the semigroup $Sing_{2\times 2}R$ of singular real 2×2 matrices. We let $Sing^*_{2\times 2}R$ be the subset of those matrices, whose leading element (i.e., that in the (1, 1)-position) is non-zero. Example 1 is to show a regular non-orthodox semigroups with a

multiplicative orthodox transversal, and the orthodox transversal is not an inverse transversal. Example 2 is to show an orthodox semigroup with a multiplicative orthodox transversal, and the orthodox transversal is neither the orthodox semigroup itself, nor the inverse transversal.

Example 1 [non-orthodox]**.** Let F be a field of characteristic 2 and consider the subset S of $Mat_{2\times 2}F$ given by

$$S = \left\{\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\} \equiv \{a, b, c, d, e\}.$$

The only non-idempotent of S is a, and clearly, $d \in V(a)$. Hence S is regular. That S is not orthodox follows from the equality $bc = a$. It is readily verified that

$$V(a) = \{d\},\ V(b) = \{d, b\},\ V(c) = \{d, c\},\ V(d) = \{d, a, b, c\},\ V(e) = \{e\}.$$

Let $S^o = \{b, d, e\}$. Then S^o is an orthodox subsemigroup of S. It is easy to check that S^o is an orthodox (in fact, band) transversal of S. Simple calculation reveal that $I = \{b, d, e\}$ and $\Lambda = \{b, c, d, e\}$. It follows that $\Lambda I = \{b, d, e\}$ and so S^o is multiplicative.

Example 2 [orthodox]**.** Let S_1 be the subset of $Sing^*_{2\times 2}R$ described by

$$S_1 = \left\{\begin{bmatrix}x & y\\ 0 & 0\end{bmatrix};\quad x, y \in R,\quad x \neq 0\right\}.$$

Then S_1 is a semigroup. It is readily seen that $\begin{bmatrix}x^{-1} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}x^{-1} & x^{-1}\\ 0 & 0\end{bmatrix} \in V$ $\begin{bmatrix}x & y\\ 0 & 0\end{bmatrix}$, and so S_1 is regular. The idempotents are the matrices of the form $\begin{bmatrix}1 & y\\ 0 & 0\end{bmatrix}$, and consequently S_1 is orthodox. Let

$$S_1^0 = \left\{\begin{bmatrix}x & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}x & x\\ 0 & 0\end{bmatrix}, x \neq 0\right\}.$$

Then S_1^0 is an orthodox subsemigroup of S and

$$V\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \cap S_1^0 = \left\{\begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x^{-1} & x^{-1} \\ 0 & 0 \end{bmatrix}\right\}.$$

It is easy to check that S_1^0 is an orthodox transversal of S. Again as can readily be verified, we have

$$I = \{XX^0;\ X \in S_1\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\} = E(S_1^0),$$

$$\Lambda = \{X^0X;\ X \in S_1\} = \left\{\begin{bmatrix} 1 & z \\ 0 & 0 \end{bmatrix};\ z \in R\right\} = E(S_1),$$

from which it follows that $\Lambda I = E(S_1^0)$, and therefore, S_1^0 is multiplicative.

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